九十六學年度高雄市高級中學數學科能力競賽試題（一）

【參考解答】

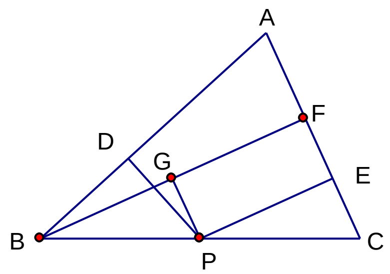
1.【參考解答】

BF⊥AC， find midpoint G of segment BF。

GP // AC。

PE⊥AC and PD⊥AB。

Then PE×PD is maximum.



【證】 Let x be PE, and y be PD.

Let b and c be the lengths of sides AC and AB, respectively.

x*b+y*c = 2d, where d is the area of triangle ABC。

y = (2d – x*b)/c。

$xy = (2dx - bx^2)/c = 1/c*[-b(x^2 - 2d/b\ x + d^2/b^2) + d^2/b]$

$= 1/c*[\ d^2/b - b(x - d/b)^2]$

xy is maximum when x = d/b。

Since area d = bh/2, when h is the height。 Hence, x=h/2。

2.【參考解答】

f(n+3) – 4f(n+2) +5f(n+1) – 2f(n) = 0

[f(n+3) – f(n+2)] – 3[f(n+2) – f(n+1)] + 2[f(n+1) – f(n)] = 0, n = 1, 2, 3, … (1)

Let g(n) = f(n+1) – f(n), n=1, 2, 3, …

g(1) = f(2) – f(1) = 0, g(2) = f(3) – f(2) = 2.

By (1), we have that g(n+2) – 3g(n+1) + 2 g(n) = 0.

Or g(n+2) – g(n+1) = 2(g(n+1) – g(n)) (2)

Let h(n) = g(n+1) – g(n), n=1, 2, 3, …

We have that h(1) = g(2) – g(1) = 2.

By (2), we have that h(n+1) = 2h(n), n = 1, 2, 3, …

Hence, we have that h(n) = 2^n.

By h(n) = g(n+1) – g(n), we have that g(n+1) – g(n) = h(n) = 2^n.

Hence g(n+1) = g(n) + 2^n, for n = 1, 2, 3, …

Therefore g(n+1) = g(1) + 2^1 + … + 2^n = 0 + 2^1 + … + 2^n = 2^{n+1} – 2.

Or g(n) = 2^n – 2, n=1, 2, 3, …

By g(n) = f(n+1) – f(n), n=1, 2, 3, …,

we have f(n+1) – f(n) = 2^n – 2, n=1, 2, 3, …

f(n+1) = f(n) + 2^n – 2 = f(n-1) +2^{n-1} – 2+ 2^n – 2 = f(n-1) +[2^{n-1} + 2^n] – [2+2]

= f(1) + [2^1 +…+ 2^n] –2n = 1 +2^{n+1} – 2 – 2n = 2^{n+1}– 2n – 1

Or f(n) = 2^n – 2(n–1) – 1 = 2^n – 2n + 1

3.【參考解答】

$\sin\alpha\sin(\alpha+2\gamma)=\sin(\pi-\alpha)\sin\left[\pi-(\alpha+2\gamma)\right]=\sin(\beta+\gamma)\sin(\beta-\gamma)$

$=(\sin\beta\cos\gamma+\cos\beta\sin\gamma)(\sin\beta\cos\gamma-\cos\beta\sin\gamma)$

$=\sin^2\beta\cos^2\gamma-\cos^2\beta\sin^2\gamma$

$=\sin^2\beta(1-\sin^2\gamma)-\cos^2\beta\sin^2\gamma$

$=\sin^2\beta-\sin^2\beta\sin^2\gamma-\cos^2\beta\sin^2\gamma$

$=\sin^2\beta-(\sin^2\beta+\cos^2\beta)\sin^2\gamma$

$=\sin^2\beta-\sin^2\gamma$ ……………………………… （1）

同理，$\sin\beta\sin(\beta+2\alpha)=\sin^2\gamma-\sin^2\alpha$ ……… （2）

$\sin\gamma\sin(\gamma+2\beta)=\sin^2\alpha-\sin^2\beta$ ………… （3）

（1）（2）（3）$\Rightarrow \sin\alpha\sin(\alpha+2\gamma)+\sin\beta\sin(\beta+2\alpha)+\sin\gamma\sin(\gamma+2\beta)=0$ #

4.【參考解答】

令 $[x]=n$ ，$x-[x]=p$ ，$0\le p<1$ 。

∵ $x=p+n$ ，$\therefore x^2-\left[x^2\right]=(n+p)^2-\left[(n+p)^2\right]=p^2$

展開整理得 $2np = \left[2np + p^2\right]$

知 $2np$ 必是整數，所以 p 只能取值 0、$\frac{1}{2n}$、$\frac{2}{2n}$、$\frac{3}{2n}$、…、$\frac{2n-1}{2n}$

若 $n \le x < n+1$，則 x 只能是：

$n+0$，$n+\frac{1}{2n}$，$n+\frac{2}{2n}$，$n+\frac{3}{2n}$，…，$n+\frac{2n-1}{2n}$ 這 $2n$ 個解。

所以 $1 \le x \le 10$，總共有 $2+4+6+8+...+18+1 = 91$ 個解。

5.【參考解答】

$\because \frac{\Delta A\ B\ H}{\Delta C\ A\ H} = \frac{{}^2\ a\ {}^2\ b}{{}^2\ a\ {}^2\ b}$ 及 $\frac{\Delta ABH}{\Delta BCH} = \frac{b^2+c^2-a^2}{a^2+b^2-c^2}$

$\because \Delta A\ B\ H = \Delta\ B\ C\ H\ \Delta$

得 $a^2-b^2+c^2 = a^2+b^2-c^2 = b^2+c^2-a^2$

解得 $a=b=c$ $\therefore \Delta ABC$ 為等邊三角形

故 $\cos A + \cos B + \cos C = \frac{3}{2}$。