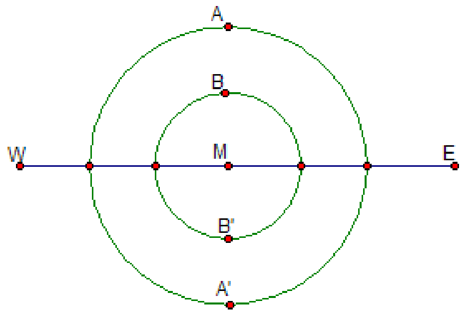
九十六學年度高級中學數學科能力競賽試題（二）

南區（高屏區）【參考解答】

一、【參考解答】

$$\left\lfloor \frac{1998!+1995!}{1997!+1996!} \right\rfloor = \left\lfloor \frac{(1998\cdot 1997+\frac{1}{1996})\cdot 1996!}{(1997+1)\cdot 1996!} \right\rfloor = \left\lfloor \frac{(1998\cdot 1997+\frac{1}{1996})}{1998} \right\rfloor = 1997 .$$

二、【參考解答】



甲、乙在 B、B'相遇機率為 $\left(\frac{1}{3}\right)^2$，甲、乙在 A、M、A'相遇機率均為 $\left(\frac{1}{3}\right)^4$，

所以兩圓時二人相遇機率為 $P_2 = 2\cdot\left(\frac{1}{3}\right)^2 + 3\cdot\left(\frac{1}{3}\right)^4 = \frac{7}{27}$，故不相遇機率為 $\frac{20}{27}$

有 n 個圓時，相遇機率為

$$\begin{aligned}
P_n &= 2\cdot\left(\frac{1}{3}\right)^2 + 2\cdot\left(\frac{1}{3}\right)^4 + \cdots + 2\cdot\left(\frac{1}{3}\right)^{2n-2} + 3\cdot\left(\frac{1}{3}\right)^{2n} \\
&= 2\cdot\left[\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^4 + \cdots + \left(\frac{1}{3}\right)^{2n}\right] + \left(\frac{1}{3}\right)^{2n} \\
&= 2\cdot\frac{\left(\frac{1}{3}\right)^2\cdot\left[1-\left(\frac{1}{3}\right)^{2n}\right]}{1-\left(\frac{1}{3}\right)^2} + \left(\frac{1}{3}\right)^{2n} \\
&= \frac{1}{4} + \frac{3}{4}\cdot\left(\frac{1}{3}\right)^{2n}
\end{aligned}$$

故不相遇機率為

$$1-\left(\frac{1}{4}+\frac{3}{4}\cdot\left(\frac{1}{3}\right)^{2n}\right)$$
$$=\frac{3}{4}-\frac{3}{4}\cdot\left(\frac{1}{3}\right)^{2n}$$

三、【參考解答】

By the diagram we know that the ratio of the length of the radii of C1 and C2 is 3:1.

Hence the ratio of the areas of C! and C2 is 9:1.

The radius of C1 is $\sqrt{}$3/6 a.

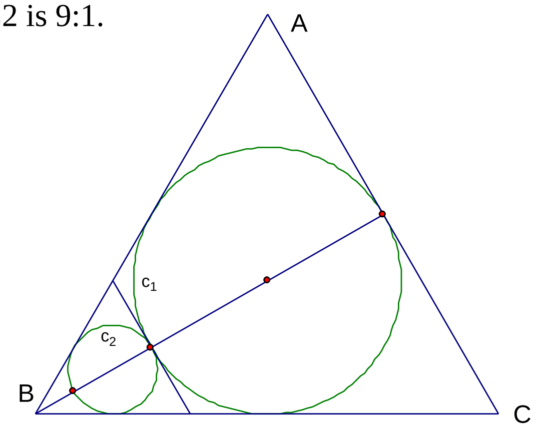
The area of C1 is π/12 a*a.

The sum of the series

1/9 + (1/9)*(1/9) + … = 1/8.

Hence the total area of the disks is

11 π/96 a*a.



四、【參考解答】

Let x=cos ^{2} t, y=sin ^{2} t, 0<t< π/2.

We know that $a^2 + b^2 \geqq (a+b)^2/2$.

Therefore f(x, y) = (cos ^{2}t + 1/(sin ^{2} t))^2 + (sin ^{2}t + 1/(cos ^{2} t))^2

≧[cos ^{2}t + 1/(sin ^{2} t) + sin ^{2}t + 1/(cos ^{2} t)]^2/2

= [(cos ^{2}t + sin ^{2}t) + (1/(sin ^{2} t) +1/(cos ^{2} t))]^2/2

= [1 + (cos ^{2}t + sin ^{2}t)/(sin ^{2} t* cos ^{2} t)]^2/2

= [1 + 4/ sin ^{2} 2t]^2/2

≧ (1 + 4)^2/2

= 25/2 When t = π/4, x = y = 1/2, min(f) = 25/2.

五、【參考解答】

注意: 假如 $A=\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}$

$A^n=\begin{bmatrix}\cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta\end{bmatrix}$

A^2

$$=\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$A^3 = AA^2 = AI_2 = A$

$A^4 = AA^3 = AA = I_2$

$A^5 = AA^4 = AI_2 = A$

$\cdots$

$$\Rightarrow A^n = \begin{cases} A & \text{，}n\text{為奇數時} \\ I_2 & \text{，}n\text{為偶數時} \end{cases}$$

$$\sum_{n=1}^{200} A^n$$

$=(A+A^3+A^5+\cdots+A^{199})+(A^2+A^4+A^6+\cdots+A^{200})$

(各100個)

$=(A+A+A+\cdots+A)+(I_2+I_2+I_2+\cdots+I_2)$

(各100個)

$=100A+100I_2$

$$=100\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}+100\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}100(1+\cos\theta) & 100\sin\theta\\ 100\sin\theta & 100(1-\cos\theta)\end{bmatrix}$$

六、【參考解答】

$$\frac{\cos 2A}{a^2}-\frac{\cos 2B}{b^2}=\frac{1-2\sin^2 A}{a^2}-\frac{1-2\sin^2 B}{b^2}=\frac{1}{a^2}-\frac{1}{b^2}-\frac{2\sin^2 A}{a^2}+\frac{2\sin^2 B}{b^2}$$

$$=\frac{1}{a^2}-\frac{1}{b^2}.$$