八十八學年度高級中學數學能力競賽(一)參考解答(台南一中

1.參考解答

（i） $f(x)=x^3+3x^2-9x+5$

（ii） 過 $A(0,5)$ 之切線方程式 $y=-9x+5$ 交 $y=f(x)$ 於 $B(-3,32)$

$P(a,f(a))$ 對切線 $y=-9x+5$ 的垂直距離

$$\ell(a)=\frac{|9a+f(a)-5|}{\sqrt{82}}=\frac{|a^3+3a^2|}{\sqrt{82}}=\frac{a^3+3a^2}{\sqrt{82}}\ (\because -3\le a\le 0)$$

$\ell'(a)=\frac{1}{\sqrt{82}}(3a^2+6a)$ 在 $a=-2$ 處，$\ell(a)$ 有最大值 $\frac{4}{\sqrt{82}}$ #

2.參考解答

$\sin\alpha=\cot\beta$ (1) $\sin\beta=\cot\gamma$ (2)

$\cot\alpha=\sin\gamma$ (3)

(1)× (3) $\cos\alpha=\cot\beta\sin\gamma$ (4)

$\sin\alpha=\cot\beta$ (1)

$(4)^2+(1)^2$: $1=\cot^2\beta(1+\sin^2\gamma)=(1+\sin^2\gamma)(\csc^2\beta-1)$

$$=(1+\sin^2\gamma)(\frac{1}{\sin^2\beta}-1)=(1+\sin^2\gamma)(\tan^2\gamma-1)$$

$$=(1+\sin^2\gamma)(\frac{\sin^2\gamma}{1-\sin^2\gamma}-1)\Rightarrow \sin^4\gamma+\sin^2\gamma-1=0$$

$$\therefore \sin^2\gamma=\frac{\sqrt{5}-1}{2}\qquad (\because \frac{\sqrt{5}+1}{2}>1)$$

$$\therefore \cot^2\alpha=\sin^2\gamma=\frac{\sqrt{5}-1}{2}=\frac{\cos^2\alpha}{1-\cos^2\alpha}\qquad \therefore \cos^2\alpha=\frac{(\sqrt{5}-1)^2}{4}$$

$\because \cos\alpha>0$ $\therefore \cos\alpha=\frac{\sqrt{5}-1}{2}$ #

3.參考解答

連 $\overline{EF}$ ，則 C、D、E、F 四點共圓

$\angle EFC = 180^\circ - \angle EDC = \angle DAB$

$\therefore$A、B、F、E 也是四點共圓

故 $\angle AEB = \angle AFB$

又 $\because \angle CED = \angle CFD$

$\therefore \angle BEC = 180^\circ - \angle AEB - \angle CED = 180^\circ - \angle AFB - \angle CFD = \angle AFD$

4.參考解答

$\because n+1 = \sqrt{(n+1)^2}$

$= \sqrt{(n+3)+(n-1)(n+2)}$

$n+2 = \sqrt{(n+4)+n(n+3)}$

$n+3 = \sqrt{(n+5)+(n+1)(n+4)}$

$\therefore n+1 = \sqrt{(n+3)+(n-1)\sqrt{(n+4)+n\sqrt{(n+5)\cdots}}}$ (*)

令 $n=3$ 代入(*) 便得證