86年台北市複賽　試題(一)參考解答

1、Let $f(x)=x^3+ax^2+bx-8=(x-x_1)(x-x_2)(x-x_3)$. Then $x_i>0$ and $x_1x_2x_3=8$. Since $2+x_i \ge 2\sqrt{2x_i}, \forall i=1,2,3$, we have

$$4a-2b-16=f(-2)=(-2-x_1)(-2-x_2)(-2-x_3)\le -8\sqrt{(2x_1)(2x_2)(2x_3)}=-64$$

It follows that $b-2a\ge 24$. The minimum of $b-2a$ can be attained by the example $f(x)=(x-2)^3=x^3-6x^2+12x-8$.

2、證明：

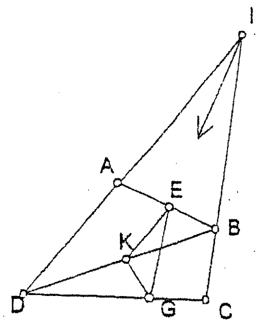
1^0.設$\overrightarrow{DA},\overrightarrow{BC}$交于$I$，

在$\overline{DB}$上取K，使$\overline{EK} // \overline{AD}$

聯$\overline{GK}$，則$\overline{GK} // \overline{BC}$

$\left(\because \dfrac{DG}{GC}=\dfrac{AE}{EB}=\dfrac{DK}{KB}\right)$

且$\dfrac{EK}{AD}=\dfrac{BE}{AB}, \dfrac{GK}{BC}=\dfrac{DG}{CD}$



$$EK=\frac{BE}{AE+BE}\cdot AD=\frac{1}{\frac{AE}{BE}+1}\cdot AD=\frac{1}{\frac{AD}{BC}+1}\cdot AD=\frac{1}{\frac{1}{BC}+\frac{1}{AD}}$$

$$GK=\frac{DG}{DG+GC}\cdot BC=\frac{1}{1+\frac{GC}{DG}}\cdot BC=\frac{1}{1+\frac{AC}{AD}}\cdot BC=\frac{1}{\frac{1}{BC}+\frac{1}{AD}}$$

$\therefore EK=GK$

作$\overrightarrow{IX} // \overrightarrow{GE}$則$\angle KEG=\angle AIX$

$\angle KGE=\angle XIB$

$\Rightarrow \overrightarrow{IX}$為$\angle AIB$之平分線

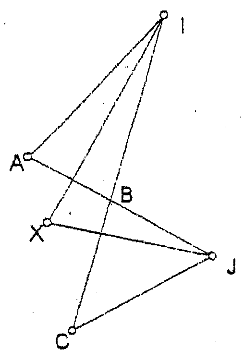
$\therefore \overline{GE} // \angle AIB$之平分線

同理$\therefore \overline{FH} // \angle\overrightarrow{AB}$為$\overrightarrow{CD}$交角之平分線

2^0.如右圖　$\overline{IX}$平分$\angle AIB$

$\overline{JX}$平分$\angle BJC$

$\Rightarrow \angle IXJ=\dfrac{1}{2}\angle A+\angle C$



3^0.由2^0 $\overrightarrow{AB}$與$\overrightarrow{CD}$交角之平分線$\perp\overrightarrow{AD}$與$\overrightarrow{BC}$交角之平分線

(即$\angle AIB$)　　　　(即$\angle BJC$)

($\because A,B,C,D$共圓，$\therefore \angle IAB$與$\angle BJC$互補)

由1^0.　$\therefore \overline{GE}\perp\overline{FH}$

3、Without loss of generality, we let $x_1 \le x_2 \le \cdots \le x_n$.

(a) Since

$$(x_1 + x_2) + (x_1 +_3) + (x_2 + x_3) = 2(x_1 + x_2 + x_3) > 0,$$

at least one of $(x_1 + x_2), (x_1 + x_3), (x_2 + x_3)$ is positive.

(b) Assume that $f(5) \le 2$. Since $b_{35} = x_3 + x_5$ and $b_{45} = x_4 + x_5$ are the biggest two terms, we have $x_2 + x_5 < 0, x_3 + x_4 < 0$. This implies $x_1 > 0$, by the inequality $x_1 + x_2 + x_3 + x_4 + x_5 > 0$. On the other hand, $x_1 \le \frac{1}{2}(x_2 + x_5) < 0$. A contradiction! Hence, $f(5) \ge 3$.

(c) The minimum of $f(1997)$ is 1996. First, using the data $x \ = x_2 = \cdots = x_{n-1} = -1, x_n = n$, we have $f(n) = n-1$. We next show by mathematical induction that

$$f(n) \ge n-1, \text{ for any odd integer } n \ge 7$$

Assume that $f(7) \le 5$. Then $b_{17} < 0$, since $b_{17} \le b_{27} \le \cdots \le b_{67}$. Similarly, $b_{36} < 0, b_{45} < 0$. This implies that $x_2 > 0$, since $x_2 + b_{17} + b_{36} + b_{45} > 0$. But this leads to a contradiction with $b_{36} = x_3 + x_6 \ge 2x_2 > 0$. Hence, $f(7) \ge 6$. Now, suppose that $f(n) \ge n-1$, we need to prove $f(n+2) \ge n+1$.

Case (i) : If $b_{1(n+2)} > 0$, then the terms $b_{i(n+2)} > 0, \forall_i = 1,2,\cdots n+1$. Thus, $f(n+2) \ge n+1$.

Case (ii) : If $b_{1(n+2)} < 0$, then $x_2 + x_3 + \cdots + x_{n+1} > 0$. By induction, there are at least $n-1$ positive sums among $x_2, x_3, \cdots, x_{n+1}$. This yields the sums $b_{n(n+2)}, b_{(n+1)(n+2)}$ are also positive. It follows that $f(n+2) \ge n+1$.