# 九十三學年度高級中學數學科能力競賽決賽

## 筆試（一）【參考解答】

[問題一] 參考解答：

連$PB$，取中點$M$，則$EM // AB$且$EM = \frac{1}{2}AB$，

連$QC$，取中點$N$，則$FN // BC$且$FN = \frac{1}{2}BC$

又$FM // PQ$且$FM = \frac{1}{2}PQ$，$GN // QR = \frac{1}{2}QR$

由$AB \perp BC$ $\therefore$ $EM \perp FN$ 而有$\angle EMF = \angle FNG$

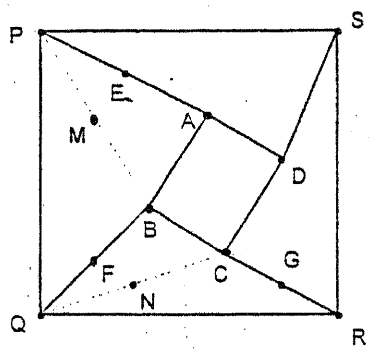
$\Rightarrow \Delta EMF \cong \Delta FNG$ $\therefore EF = FG$

延長$MF$，$NG$交於$K$，則有

$\angle EFG = \angle MFG - \angle MFE$ 又 $\angle MFG = \angle K + \angle KGF$

但$\angle K = 90°$且$\angle MFG = \angle KGF$ $\therefore$ $\angle EFG = 90°$

同理可證：$FG = GH$且$\angle FGH = 90°$，因此四邊形$EFGH$為正方形




[問題二] 參考解答：

證明 ：

令$x = \sqrt[k]{a_1 \cdot a_2 \cdots a_k}$ 則 $1 > a_1 \cdot a_2 \cdots a_k \cdot a_{k+1} \geq x^{k+1} > 0$

則$f(k+1) = \dfrac{k+1}{1 - a_1 \cdot a_2 \cdots a_k \cdot a_{k+1}} \geq \dfrac{k+1}{1-x^{k+1}} = \dfrac{k+1}{(1-x)(1+x+x^2+\cdots+x^k)}$

$\because 1 > x > 0 \Rightarrow \dfrac{1+x+x^2+\cdots+x^k}{k+1} < \dfrac{1+x+x^2+\cdots+x^{k-1}}{k}$

$$\therefore\ f(k+1) = \frac{1}{(1-x)\dfrac{1+x+x^2+\cdots+x^k}{k+1}}$$

$$> \frac{1}{(1-x)\dfrac{1+x+x^2+\cdots+x^{k-1}}{k}} = \frac{k}{1-x^k} = f(k)$$

[問題三] 參考解答：

不妨設 $a_1 < a_2 < \cdots < a_n$.

先用反證法證明對任意正整數$k \le n$，$\sum_{i=1}^{k} a_i \ge 2^k - 1$.

若$\sum_{i=1}^{k} a_i < 2^k - 1$,則$\{a_1, a_2, \cdots, a_k\}$的每個非空子集合的元素和不超過$2^k - 2$,

但 $\{a_1, a_2, \cdots, a_k\}$ 有 $2^k - 1$ 個非空子集，按抽屜原則，必有兩個非空子集的元素和相等，這與題設矛盾。

$$1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} - \left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right) = \frac{a_1 - 1}{a_1} + \frac{a_2 - 2}{2a_2} + \cdots + \frac{a_n - 2^{n-1}}{2^{n-1} a_n}.$$

令 $c_i = \frac{1}{2^{i-1} a_i}$，$d_i = a_i - 2^{i-1}$，$D_k = \sum_{i=1}^{k} d_i$，則 $c_1 > c_2 > \cdots > c_n$，

且 $D_k = \sum_{i=1}^{k} a_i - (1 + 2 + \cdots + 2^{k-1}) = \sum_{i=1}^{k} a_i - (2^k - 1) \ge 0$．於是

$$\begin{aligned}
&1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} - \left(\frac{1}{a_1} + \cdots + \frac{1}{a_n}\right) = \sum_{i=1}^{n} c_i d_i \\
&= c_1 D_1 + c_2 (D_2 - D_1) + \cdots + c_n (D_n - D_{n-1}) \\
&= (c_1 - c_2) D_1 + (c_2 - c_3) D_2 + \cdots + (c_{n-1} - c_n) D_{n-1} + c_n D_n \\
&\ge 0
\end{aligned}$$

這就證明了

$$\frac{1}{a_1} + \cdots + \frac{1}{a_n} \le 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}$$

容易驗證集合$S = \{1, 2, 2^2, \cdots, 2^{n-1}\}$滿足題設條件,且

$$1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}$$

綜上所述,所求最大值是$2 - \frac{1}{2^{n-1}}$