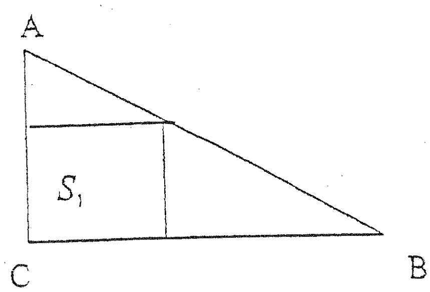
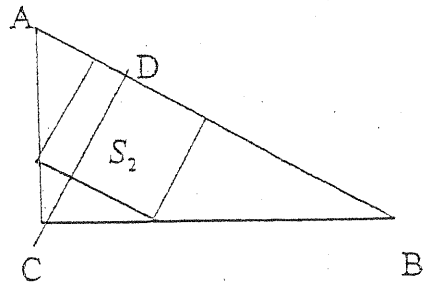
【筆試問題四參考解答】





令 $\overline{AC}=a+19$, $\overline{BC}=b+19$, $\overline{AB}=c$

過 C 作 $\overline{CD}\perp\overline{AB}$ 於 D， S_2 邊長為 x

由 △ 相似性質知 $\dfrac{a}{19}=\dfrac{19}{b}\Rightarrow ab=19^2$

且由 $a+b=380-38=38\times 9$

$$\therefore c^2=(a+19)^2+(b+19)^2=(a+b)^2-2ab+38(a+b)+19^2\times 2$$
$$=38^2\times 9^2-2\times 19^2+38^2\times 9+2\times 19^2=38^2\times 9\times 10$$

$\Rightarrow c=38\times 3\sqrt{10}=114\sqrt{10}$

$$\therefore \overline{CD}=\frac{(a+19)(b+19)}{c}=\frac{19^2(2+18)}{114\sqrt{10}}=\frac{19\sqrt{10}}{3}$$

由 △ 相似性質 $\dfrac{\dfrac{19\sqrt{10}}{3}-x}{x}=\dfrac{\dfrac{19\sqrt{10}}{3}}{114\sqrt{10}}=\dfrac{1}{18}$

$x=114\sqrt{10}-18x$

$19x=114\sqrt{10}$

$x=6\sqrt{10}$

$\therefore x^2=36\times 10=360$　　　$\therefore S_2$ 的面積 $=360cm^2$

【筆試問題五參考解答】

$\because a_{n+1} = ca_n + \sqrt{(c^2-1)(a_n^2-4)}$

$\therefore (a_{n+1} - ca_n) = \left(\sqrt{(c^2-1)(a_n^2-4)}\right)^2$

即$a_{n+1}^2 + c^2a_n^2 - 2ca_na_{n+1} = c^2a_n^2 - 4c^2 - a_n^2 + 4$

$\Rightarrow a_{n+1}^2 - 2ca_na_{n+1} + 4c^2 + a_n^2 - 4 = 0 \cdots\cdots(1)$

又$(a_{n+2} - ca_{n+1})^2 = \left(\sqrt{(c^2-1)(a_{n+1}^2-4)}\right)^2$

$\therefore a_{n+2}^2 + c^2a_{n+1}^2 - 2ca_{n+1}a_{n+2} = c^2a_{n+1}^2 - 4c^2 - a_{n+1}^2 + 4$

即$a_{n+2}^2 - 2ca_{n+1}a_{n+2} + 4c^2 + a_{n+1}^2 - 4 = 0 \cdots\cdots(2)$

$(2)-(1) \Rightarrow a_{n+2}^2 - 2ca_{n+1}a_{n+2} + 2ca_na_{n+1} - a_n^2 = 0$

即$(a_{n+2} - a_n)(a_{n+2} + a_n) - 2ca_{n+1}(a_{n+2} - a_n) = 0$

即$(a_{n+2} - a_n)(a_{n+2} + a_n - 2a_{n+1}c) = 0$

又$a_{n+2} - a_n \neq 0 \therefore a_{n+2} + a_n - 2ca_{n+1} = 0$

即$a_{n+2} = 2ca_{n+1} - a_n$

$\because a_1 = 2 \therefore a_2 = ca_1 + \sqrt{(c^2-1)(a_1^2-4)} = 2c \in Z$($\because c$爲非零之整數)

由數學歸納法可將通式寫成

$a_{n+2} = 2ca_{n+1} - a_n$,其中$a_{n+2}$的係數爲 1,而$a_{n+1}$係數爲$2c \in Z$,$a_n$爲 1

$\therefore \forall n \in N$，$a_n \in Z \therefore a_{2000}$爲整數

【筆試問題六參考解答】

利用數學歸納法

$n = 1$時,$\dfrac{a_1 + a_3}{2} \leq a_2$成立

設$n = k$時成立,即$\dfrac{a_1 + a_3 + \cdots + a_{2k+1}}{k+1} \leq \dfrac{a_2 + a_4 + \cdots + a_{2k}}{k}$,

即$k(a_1 + a_2 + \cdots + a_{2k+1}) \leq (k+1)(a_2 + a_4 + \cdots + a_{2k}) \cdots\cdots(1)$

設$n = k+1$時,欲證$(k+1)(a_1 + a_2 + \cdots + a_{2k+3}) \leq (k+2)(a_2 + a_4 + \cdots + a_{2k+2})$

由題意知:$a_n + a_{n+2} \leq 2a_{n+1}$即$a_{n+2} - a_{n+1} \leq a_{n+1} - a_n$

$\therefore$ 數列$\{a_{n+1} - a_n\}$爲遞減

$(k+1)(a_{2k+3} - a_{2k+2}) \leq (a_2 - a_1) + (a_4 - a_3) + \cdots + (a_{2k+2} - a_{2k+1})$

即$a_1 + a_3 + \cdots + a_{2k+1} + (k+1)a_{2k+3} \leq a_2 + a_4 + \cdots + a_{2k} + (k+2)a_{2k+2} \cdots\cdots(2)$

$(1)+(2)$ $\quad (k+1)(a_1 + a_3 + \cdots + a_{2k+3}) \leq (k+2)(a_2 + a_4 + \cdots + a_{2k} + a_{2k+2})$

即$\dfrac{a_1 + a_3 + \cdots + a_{2(k+1)+1}}{(k+1)+1} \leq \dfrac{a_2 + a_4 + \cdots + a_{2(k+1)}}{k+1}$ 故得證